

FINITENESS OF HOMOTOPY TYPES OF RIGHT ORBITS OF MORSE FUNCTIONS ON SURFACES

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ABSTRACT. Let M be a connected orientable surface, P be either the real line \mathbb{R} or the circle S^1 , and $f : M \rightarrow P$ be a Morse map. Denote by \mathcal{D}_{id} the group of diffeomorphisms of M isotopic to the identity. This group acts from the right on the space of smooth maps $C^\infty(M, P)$ and one can define the stabilizer $\mathcal{S} = \{h \in \mathcal{D}_{\text{id}} \mid f \circ h = f\}$ and the orbit $\mathcal{O} = \{f \circ h \mid h \in \mathcal{D}_{\text{id}}\}$ of f with respect to that action.

In turn, the stabilizer \mathcal{S} acts on the Kronrod-Reeb graph Γ of f . Denote by G the group of all automorphisms of Γ induced by elements from \mathcal{S} .

It is proved that if M is distinct from the 2-sphere and 2-torus then there exists a free action of G on a p -dimensional torus T^p for some $p \geq 0$ such that the orbit \mathcal{O} (endowed with C^∞ topology) is homotopy equivalent to the factor space T^p/G .

1. INTRODUCTION

Let M be a compact connected surface and P be either the real line \mathbb{R} or the circle S^1 . For a closed subset $V \subset M$ denote by $\mathcal{D}(M, V)$ the group of all C^∞ diffeomorphisms of M fixed on V . This group acts from the right on the space $C^\infty(M, P)$ by the following rule: if $h \in \mathcal{D}(M, V)$ and $f \in C^\infty(M, P)$, then the result of the action of h on f is the composition map $f \circ h : M \rightarrow P$.

For $f \in C^\infty(M, P)$ let

$$\mathcal{S}(f, V) = \{h \in \mathcal{D}(M, V) \mid f \circ h = f\}, \quad \mathcal{O}(f, V) = \{f \circ h \mid h \in \mathcal{D}(M, V)\}$$

be respectively the *stabilizer* and the *orbit* of f under that action. Endow these spaces with C^∞ topologies and denote by $\mathcal{D}_{\text{id}}(M, V)$ and $\mathcal{S}_{\text{id}}(f, V)$ the corresponding path components of id_M in $\mathcal{D}(M, V)$ and $\mathcal{S}(f, V)$, and by $\mathcal{O}_f(f, V)$ the path component of $\mathcal{O}(f, V)$ containing f . We will omit V from notation whenever it is empty.

Definition 1.1. A smooth map $f : M \rightarrow P$ will be called *Morse* if

- all critical points of f are non-degenerate and belong to the interior of M ;
- the restriction of f to each connected component of ∂M is a constant map.

In a series of papers [4, 5, 7, 8] the author studied the homotopy types of $\mathcal{S}_{\text{id}}(f)$ and $\mathcal{O}_f(f)$ for certain classes of smooth mappings $f : M \rightarrow P$. The results concerning Morse maps can be summarized as follows.

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Theorem 1.2. [4, 5, 7, 8]. *Let $f : M \rightarrow P$ be a Morse map with n critical points, and V be a finite (possibly empty) union of regular (that is containing no critical points) connected components of certain level sets of f . Then the following statements hold.*

(1) $\mathcal{O}_f(f, V) = \mathcal{O}_f(f, V \cup \partial M)$ and this space **has a homotopy type of some (possibly non-compact) $(2n - 1)$ -dimensional CW-complex**.

(2) The map $\sigma : \mathcal{D}(M, V) \rightarrow \mathcal{O}(f, V)$ defined by $\sigma(h) = f \circ h$ is a Serre fibration with fiber $\mathcal{S}(f, V)$, i.e. it has a homotopy lifting property for CW-complexes.

Suppose either f has a critical point of index 1 or M is non-orientable. Then $\mathcal{S}_{\text{id}}(f)$ is contractible, $\pi_k \mathcal{O}(f) = \pi_k M$ for $k \geq 3$, $\pi_2 \mathcal{O}(f) = 0$, and for $\pi_1 \mathcal{O}(f)$ we have the following short exact sequence of fibration σ :

$$(1.1) \quad 1 \rightarrow \pi_1 \mathcal{D}(M) \xrightarrow{\sigma} \pi_1 \mathcal{O}(f) \xrightarrow{\partial} \pi_0 \mathcal{S}'(f) \rightarrow 1,$$

where $\mathcal{S}'(f) = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(M)$. If f is **generic**, i.e. each critical level set of f contains exactly one critical point, then $\mathcal{O}_f(f)$ is **homotopy equivalent to some p -dimensional torus T^p** .

(3) The restriction $\sigma|_{\mathcal{D}_{\text{id}}(M, V)} : \mathcal{D}_{\text{id}}(M, V) \rightarrow \mathcal{O}_f(f, V)$ is also a Serre fibration with fiber $\mathcal{S}'(f, V) = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(M, V)$.

Suppose either $\chi(M) < 0$ or $V \neq \emptyset$. Then $\mathcal{D}_{\text{id}}(M, V)$ and $\mathcal{S}_{\text{id}}(f, V)$ are contractible and from the exact sequence of homotopy groups of the fibration $\sigma|_{\mathcal{D}_{\text{id}}(M, V)}$ we get that $\pi_k \mathcal{O}(f, V) = 0$ for $k \geq 2$ and the boundary map

$$(1.2) \quad \partial : \pi_1 \mathcal{O}(f, V) \rightarrow \pi_0 \mathcal{S}'(f, V)$$

is an isomorphism. □

In the present note we obtain exact description of the homotopy types of $\mathcal{O}_f(f)$ for all Morse maps $f : M \rightarrow P$ for the case when M is orientable and distinct from S^2 and T^2 , see Theorem 1.3 below.

Let $f : M \rightarrow P$ be a Morse map. Consider the partition of M into connected components of level-sets $f^{-1}(c)$, $c \in P$. Then the corresponding factor space has a structure of a finite one-dimensional CW-complex and is called the *Kronrod-Reeb* graph of f , see e.g. [1, 12]. We will denote this graph by Γ . Its vertices correspond to *critical* (i.e. containing critical points) connected components of level-sets of f and to the connected components of ∂M .

Recall, [4, §3.1], that $\mathcal{S}(f)$ naturally acts on Γ by the following rule. Let $h \in \mathcal{S}(f)$. Then $f \circ h = f$, and so $h(f^{-1}(c)) = f^{-1}(c)$ for all $c \in P$. Hence h permutes connected components of $f^{-1}(c)$ being *points* of Γ . In other words, h yields a bijection of Γ which is in fact homeomorphism. Therefore we get an action homomorphism $\lambda : \mathcal{S}(f) \rightarrow \text{Aut}(\Gamma)$ into the group of all homeomorphisms of Γ . The image of λ is finite. Let

$$G := \lambda(\mathcal{S}'(f))$$

be the group of all automorphisms of Γ induced by isotopic to the identity diffeomorphisms from $\mathcal{S}(f)$. Our main result is the following theorem.

Theorem 1.3. *Suppose M is orientable and distinct from 2-sphere S^2 and 2-torus T^2 . Then for each Morse map $f : M \rightarrow P$ there exists a free action of the group $G = \lambda(\mathcal{S}'(f))$ on a p -dimensional torus T^p for some $p \geq 0$ such that $\mathcal{O}_f(f)$ is homotopy equivalent to the factor space T^p/G .*

This result refines statements (1) and (2) of Theorem 1.2. A sketch of a proof of Theorem 1.3 will be given in §3 and §4. The proof is constructive in the sense that for every Morse map $f : M \rightarrow P$ one can explicitly describe the corresponding free action of G on T^p which gives a homotopy equivalence $T^p/G \simeq \mathcal{O}_f(f)$.

If $M = S^2$, then by (2) of Theorem 1.2 $\mathcal{O}_f(f)$ is not aspherical, and so it is not homotopy equivalent to T^p/G for any free G -action on any torus T^p . On the other hand there are some partial results supporting the conjecture that Theorem 1.3 might hold for $M = T^2$, see [11].

In fact, Theorem 1.3 is true for a larger class of maps f having singularities smoothly equivalent to homogeneous polynomials $\mathbb{R}^2 \rightarrow \mathbb{R}$ without multiple factors. But in this case one should extend the notion of Kronrod-Reeb graph for such maps. This version of Theorem 1.3 together with detailed proofs will appear elsewhere.

2. PRELIMINARIES

2.1. Wreath products. Let G and H be two groups. Then the set G^H of all maps $H \rightarrow G$ (not necessarily homomorphisms) is a group with respect to the point-wise multiplication. Moreover, the group H acts on G^H from the right by the following rule: if $\alpha : H \rightarrow G$, and $h \in H$, then the result α^h of the action of h on α is defined by the formula: $\alpha^h(s) = \alpha(sh)$, for all $s \in H$. The semidirect product $G^H \rtimes H$ associated with this action is called the *wreath product* of G and H and is denoted by $G \wr H$.

More generally, let K be another group and $\mu : K \rightarrow H$ be a homomorphism. This homomorphism induces a natural right action of K on H , and so one can define the corresponding semidirect product $G^H \rtimes_{\mu} K$ which will be called the *wreath product of G and K over μ* and denoted by $G \wr_{\mu} K$. Evidently, if $\mu = \text{id}_H : H \rightarrow H$ is the identity isomorphism, then $G \wr_{\mu} H$ is the same as $G \wr H$.

In particular, for $m \geq 2$ let $\mu : \mathbb{Z} \rightarrow \mathbb{Z}_m$ be the natural mod m homomorphism. Then for every group \mathcal{S} the wreath product $\mathcal{S} \wr_{\mu} \mathbb{Z}$ will be denoted by

$$\mathcal{S} \wr_{\mathbb{Z}_m} \mathbb{Z}.$$

Thus $\mathcal{S} \wr_{\mathbb{Z}_m} \mathbb{Z}$ is the set $\mathcal{S}^{\mathbb{Z}_m} \times \mathbb{Z}$ with the multiplication defined in the following way: if $(\alpha, a), (\beta, b) \in \mathcal{S}^{\mathbb{Z}_m} \times \mathbb{Z}$, then $(\alpha, a)(\beta, b) = (\gamma, a + b)$, where $\gamma : \mathbb{Z}_m \rightarrow \mathcal{S}$ is given by $\gamma(s) = \alpha(s + b \bmod m)\beta(s)$ for $s \in \mathbb{Z}_m$.

2.2. Free actions on tori. For $p \geq 0$ let $T^p = \mathbb{R}^p/\mathbb{Z}^p$ be a p -dimensional torus which can also be regarded as a topological product of p circles. We also assume that T^0 is a point.

Suppose a finite group G freely acts on T^p . Then the factor map $q : T^p \rightarrow T^p/G$ is a covering map. This implies that T^p/G is aspherical and we have the following short exact sequence:

$$1 \longrightarrow \pi_1 T^p \xrightarrow{q} \pi_1(T^p/G) \xrightarrow{\delta} G \longrightarrow 1,$$

where δ is the corresponding boundary homomorphism.

Definition 2.1. Let $\lambda : \mathcal{S} \rightarrow G$ be a group homomorphism. We will say that λ *arises from a free action on a torus* if there exists a free action of G on T^p for some $p \geq 0$ and the following commutative diagram:

$$\begin{array}{ccc} \pi_1(T^p/G) & \xrightarrow[\cong]{\alpha} & \mathcal{S} \\ \delta \downarrow & & \downarrow \lambda \\ G & \xrightarrow[\cong]{\beta} & G \end{array}$$

in which α and β are isomorphisms.

Lemma 2.2. (1) A trivial homomorphism $e_n : \mathbb{Z}^n \rightarrow \{1\}$, $n \geq 0$, arises from the free trivial action of the unit group $\{1\}$ on T^n .

(2) For each $m \geq 1$ the canonical $(\text{mod } m)$ -epimorphism $\mu : \mathbb{Z} \rightarrow \mathbb{Z}_m$ arises from the free action of \mathbb{Z}_m on T^1 defined as follows: if $x \in T^1 = \mathbb{R}/\mathbb{Z}$ and $k \in \mathbb{Z}_m$, then $z \cdot k = z + \frac{k}{m} \text{ mod } 1$.

(3) Suppose a homomorphism $\lambda_i : \mathcal{S}_i \rightarrow G_i$ arises from a free action of G_i on T^{p_i} for $i = 1, \dots, n$. Denote $p = \sum_{i=1}^n p_i$. Then the product homomorphism

$$\bigtimes_{i=1}^n \lambda_i : \bigtimes_{i=1}^n \mathcal{S}_i \longrightarrow \bigtimes_{i=1}^n G_i$$

arises from the free action of $\bigtimes_{i=1}^n G_i$ on $T^p = T^{p_1} \times \dots \times T^{p_n}$ defined as follows: if $x_i \in T^{p_i}$ and $g \in G_i$, then $(x_1, \dots, x_n) \cdot (g_1, \dots, g_n) = (x_1 g_1, \dots, x_n g_n)$.

(4) Suppose $\lambda : \mathcal{S} \rightarrow G$ arises from a free action of G on T^p and let $n \geq 0$. Then the homomorphism $\bar{\lambda} : \mathcal{S} \times \mathbb{Z}^n \rightarrow G$ defined by $\bar{\lambda}(s, z) = \lambda(s)$ arises from the free G -action on $T^{p+n} = T^p \times T^n$ defined as follows: if $x \in T^p$, $y \in T^n$, and $g \in G$, then $(x, y) \cdot g = (xg, y)$.

(5) Suppose $\lambda : \mathcal{S} \rightarrow G$ arises from a free action of G on T^p and let $m \geq 2$. Define the following homomorphism $\bar{\lambda} : \mathcal{S} \wr_{\mathbb{Z}_m} \mathbb{Z} \rightarrow G \wr_{\mathbb{Z}_m} \mathbb{Z}_m$ by

$$(2.1) \quad \bar{\lambda}(\xi, k) = (\lambda \circ \xi, k \text{ mod } m),$$

for $(\xi : \mathbb{Z}_m \rightarrow \mathcal{S}, k) \in \mathcal{S} \wr_{\mathbb{Z}_m} \mathbb{Z}$. Then $\bar{\lambda}$ arises from the free action of $G \wr_{\mathbb{Z}_m} \mathbb{Z}_m$ on $T^{pm+1} = \underbrace{T^p \times \dots \times T^p}_m \times T^1$ defined as follows: if $x_i \in T^p$, $i = 0, \dots, m-1$, $y \in T^1$,

and $(\alpha : \mathbb{Z}_m \rightarrow G, k) \in G \wr \mathbb{Z}_m$, then

$$(x_0, x_1, \dots, x_{m-1}, y) \cdot (\alpha, k) = \left(x_k \alpha(0), x_{k+1} \alpha(1), \dots, y + \frac{k}{m} \bmod 1 \right),$$

where all indices are taken modulo m .

Proof. Statements (1)-(4) are easy. The proof of (5) is technical and will be published elsewhere. \square

3. PROOF OF THEOREM 1.3

Suppose M is orientable and distinct from S^2 and T^2 . Let also $f : M \rightarrow P$ be a Morse map, $\lambda : \mathcal{S}(f) \rightarrow \text{Aut}(\Gamma)$ be the action homomorphism, $G = \lambda(\mathcal{S}'(f))$, and V be any (possibly empty) collection of connected components of ∂M .

Notice that each $h \in \mathcal{S}'(f)$ preserves every connected component of ∂M with its orientation, and so it is isotopic in $\mathcal{S}'(f)$ to a diffeomorphism h_1 fixed on ∂M , see e.g. [8, Lemma 6.1]. In particular, we obtain that $\lambda(h) = \lambda(h_1)$, whence

$$G = \lambda(\mathcal{S}'(f)) = \lambda(\mathcal{S}'(f, V)) = \lambda(\mathcal{S}'(f, \partial M)).$$

Moreover, it is easy to see that for each $h \in \mathcal{S}'(f, V)$ its image $\lambda(h)$ depends only on the isotopy class of $[h]$ in $\mathcal{S}'(f, V)$, and so λ induces the following epimorphism

$$(3.1) \quad \lambda : \pi_0 \mathcal{S}'(f, V) \rightarrow G.$$

Proposition 3.1. *There exists a collection V of connected components of ∂M satisfying the following conditions:*

- (a) *the boundary map $\partial : \pi_1 \mathcal{O}(f, V) \rightarrow \pi_0 \mathcal{S}'(f, V)$ is an isomorphism;*
- (b) *the homomorphism (3.1) arises from a free G -action on some torus T^p .*

It follows from (b) that we have an isomorphism $\theta : \pi_1(T^p/G) \rightarrow \pi_0 \mathcal{S}'(f, V)$, which together with (1) of Theorem 1.2 and (a) gives another isomorphism:

$$\partial^{-1} \circ \theta : \pi_1(T^p/G) \xrightarrow{\theta} \pi_0 \mathcal{S}'(f, V) \xrightarrow{\partial^{-1}} \pi_1 \mathcal{O}(f, V) \equiv \pi_1 \mathcal{O}_f(f).$$

As both T^p/G and $\mathcal{O}_f(f)$ are aspherical, there exists a homotopy equivalence between these spaces such that the corresponding isomorphism of fundamental groups coincides with $\partial^{-1} \circ \theta$. This proves Theorem 1.3 modulo Proposition 3.1.

4. PROOF OF PROPOSITION 3.1

Case $M = D^2$ or $S^1 \times I$. Let V be a connected component of ∂M and $\mathcal{S} = \mathcal{S}'(f, V)$. Then by (1.2) the boundary map $\partial : \pi_1 \mathcal{O}_f(f, V) \rightarrow \pi_0 \mathcal{S}$ is an isomorphism. Therefore it remains to show that the homomorphism $\lambda : \pi_0 \mathcal{S} \rightarrow G$ arises from a certain free G -action on a torus T^p for some $p \geq 0$.

We will use an induction on the number n of critical points of f .

Notice also that $\mathcal{D}(M, V)$ is contractible, [13, 2, 3]. Hence $\mathcal{D}(M, V) = \mathcal{D}_{\text{id}}(M, V)$ and so $\mathcal{S} = \mathcal{S}(f) \cap \mathcal{D}_{\text{id}}(M, V) = \mathcal{S}(f) \cap \mathcal{D}(M, V) = \mathcal{S}(f, V)$.

1) Suppose $n = 0$. This is possible only when $M = S^1 \times I$, see Figure 1. We can also assume that $V = S^1 \times 0$ and each of the circles $S^1 \times s$, $s \in I$, is a connected

FIGURE 1. Cases $n = 0, 1$

component of some level set of f . In this case Γ consists of a unique edge, and therefore $G = \lambda(\pi_0\mathcal{S}) = \{1\}$. It follows from the proof of Case 2 of Lemma 6.3 from [4, page 261] that $\pi_0\mathcal{S} = \{1\}$. Then by (1) of Lemma 2.2 $\lambda : \pi_0\mathcal{S} \rightarrow G$ arises from a trivial action of G on T^0 .

2) Suppose $n = 1$. In this case $M = D^2$, $V = \partial M$, a unique critical point q of f is a local extreme, and Γ again consists of a unique edge, so $G = \{1\}$, see Figure 1. By Morse lemma we have that $f(x, y) = x^2 + y^2$ in some local representation of f at q , therefore not loosing generality one can assume that $M = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is a unit 2-disk in the complex plane, and the level sets of f are the origin $0 \in \mathbb{C}$ and concentric circles $S_t = \{|z| = t\}$, $t \in (0, 1]$. It follows from the proof of Case 3 of Lemma 6.3 from [4, page 261] that $\pi_0\mathcal{S} = \{1\}$. Hence again by (1) of Lemma 2.2 $\lambda : \pi_0\mathcal{S} \rightarrow G$ arises from a trivial action of G on T^0 .

3) Now let $n > 1$. Assume that we proved our statement for all Morse maps $D^2 \rightarrow P$ and $S^1 \times I \rightarrow P$ having less than n critical points, and let $f : M \rightarrow P$ be a Morse map with exactly n critical points.

Notice that the Kronrod-Reeb graph Γ of f is a finite tree. Let v be a vertex Γ of f corresponding to V . Then there exists a unique edge $e = (v, u)$ in Γ incident to v . Consider another vertex u of e and let U be the connected component of the level set of f corresponding to u , see Figure 2.

For $\varepsilon > 0$ let J_ε be a closed ε -neighbourhood of the point $t = f(U)$ in P and N be the connected component of $f^{-1}(J_\varepsilon)$ containing U . We can choose ε so small that $N \cap \partial M = \emptyset$ and the set $N \setminus U$ contains no critical points of f .

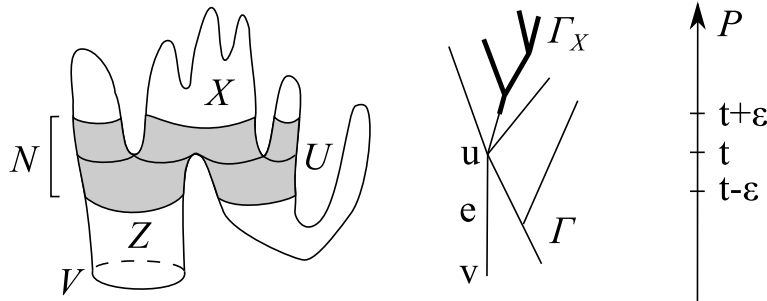


FIGURE 2.

Let Z be a connected component of $\overline{M \setminus N}$ containing V . Then Z is a cylinder having no critical points of f . Let $\tau : M \rightarrow M$ be a Dehn twist preserving f and

supported in the interior of Z , see [4, §6], and $\tau \in \pi_0\mathcal{S}$ be its isotopy class. Evidently, $\lambda(\tau) = \text{id}_\Gamma$.

Lemma 4.1. [10] *There exists an epimorphism $\eta : \pi_0\mathcal{S} \rightarrow \mathbb{Z}$ having the following properties.*

- (a) Denote $m = \eta(\tau)$. Then $m \geq 1$.
- (b) If $\mathbf{h} \in \pi_0\mathcal{S}$ is such that $\eta(\mathbf{h})$ is divided by m , then the class \mathbf{h} contains a representative fixed on some neighbourhood of N .
- (c) If $\eta(\mathbf{h}) = 0$, then \mathbf{h} contains a representative fixed on some neighbourhood of $N \cup Z$.

For each connected component X of $\overline{M \setminus N}$ put $\hat{X} := X \cap \partial N$. Then the pair (X, \hat{X}) is diffeomorphic either to $(D^2, \partial D^2)$ or to $(S^1 \times I, S^1 \times 0)$. As f is constant on connected components of ∂X , the restriction $f_X = f|_X : X \rightarrow P$ is Morse in the sense of Definition 1.1.

Let Γ_X be the Kronrod-Reeb graph of f_X . Then Γ_X can be regarded as a subtree of Γ . Denote $\mathcal{S}_X := \mathcal{S}(f_X, \hat{X})$. Let also $\lambda_X : \pi_0\mathcal{S}_X \rightarrow \text{Aut}(\Gamma_X)$ be the corresponding action homomorphism, and $G_X = \lambda_X(\pi_0\mathcal{S}_X)$.

Since the number of critical points of f_X is less than n , we have by inductive assumption that $\lambda_X : \pi_0\mathcal{S}_X \rightarrow G_X$ arises from a free action on some torus.

Now the situation splits into two cases.

3a) Suppose $m = 1$. Then by (b) of Lemma 4.1 every $\mathbf{h} \in \pi_0\mathcal{S}$ has a representative fixed near N . Hence if we denote by X_1, \dots, X_a all the connected components of $\overline{M \setminus N}$ distinct from Z , then $h(X_i) = X_i$ for all $i = 1, \dots, a$ and $h \in \mathcal{S}$.

Lemma 4.2. [9, 10] *There is the following commutative diagram:*

$$(4.1) \quad \begin{array}{ccc} \left(\bigtimes_{i=1}^a \pi_0\mathcal{S}_{X_i} \right) \times \mathbb{Z} & \xrightarrow[\cong]{\alpha} & \pi_0\mathcal{S} \\ \bar{\lambda} \downarrow & & \downarrow \lambda \\ \bigtimes_{i=1}^a G_{X_i} & \xrightarrow[\cong]{\beta} & G \end{array}$$

in which α and β are isomorphisms, and $\bar{\lambda}$ is defined by

$$\bar{\lambda}(\mathbf{h}_1, \dots, \mathbf{h}_a, k) = (\lambda_1(\mathbf{h}_1), \dots, \lambda_a(\mathbf{h}_a))$$

for $\mathbf{h}_i \in \mathcal{S}_{X_i}$ and $k \in \mathbb{Z}$. Hence by (3) and (4) of Lemma 2.2 λ arises from a free action on a torus.

Proof. Isomorphisms α and β are constructed [10] and [9] respectively. We will just recall their definitions.

Let $\mathbf{h}_i \in \pi_0\mathcal{S}_{X_i}$, $i = 1, \dots, a$, and $k \in \mathbb{Z}$. Choose a representative $h_i \in \mathcal{S}_{X_i}$ of \mathbf{h}_i such that h_i is fixed on some neighbourhood of \hat{X}_i , and so it extends by the identity to a diffeomorphism of M . Let $\mathbf{h} \in \pi_0\mathcal{S}$ be the isotopy class of the composition $h_1 \circ \dots \circ h_a$. Put $\alpha(\mathbf{h}_1, \dots, \mathbf{h}_a, k) = \mathbf{h}\tau^k$. It is shown in [10] that α is a group isomorphism.

Let $\gamma_i \in G_{X_i}$, $i = 1, \dots, a$. Notice that Γ_{X_i} is a subtree of Γ and that γ_i extends by the identity to an automorphism of all of Γ . Let $\gamma = \gamma_1 \circ \dots \circ \gamma_n$. Then it is easy to show that $\gamma \in G$, i.e. it is induced by some $h \in \mathcal{S}$, so we set $\beta(\gamma_1, \dots, \gamma_n) = \gamma$. It is proven in [9] that β is a group isomorphism. It also easily follows from the definitions that the diagram (4.1) is commutative. \square

3b) Suppose $m \geq 2$. Let X_1, \dots, X_a be all the connected components of $\overline{M \setminus N}$ distinct from Z and *invariant* with respect to all $h \in \mathcal{S}$, and Y be the collection of all connected components of $\overline{M \setminus N}$ being *not invariant* under \mathcal{S} .

Lemma 4.3. [9, 10] *There exist a subcollection $\tilde{Y} = \{Y_1, \dots, Y_b\} \subset Y$ and $g \in \mathcal{S}$ having the following properties:*

- (1) $\eta(g) = 1$ and g is fixed near each X_i , $i = 1, \dots, a$;
- (2) $g^i(\tilde{Y}) \cap g^j(\tilde{Y}) = \emptyset$ for $i \neq j \in \{0, \dots, m-1\}$;
- (3) $Y = \bigcup_{i=0}^{m-1} g^i(\tilde{Y})$;
- (4) g^m preserves every connected component of $\overline{M \setminus N}$;
- (5) $\lambda(g^m) = \text{id}_\Gamma$.

Moreover, there is the following commutative diagram:

$$(4.2) \quad \begin{array}{ccc} \left(\bigotimes_{i=1}^a \pi_0 \mathcal{S}_{X_i} \right) \times \left(\left(\bigotimes_{j=1}^b \pi_0 \mathcal{S}_{Y_j} \right) \wr_{\mathbb{Z}_m} \mathbb{Z} \right) & \xrightarrow[\cong]{\alpha} & \pi_0 \mathcal{S} \\ \bar{\lambda} \downarrow & & \downarrow \lambda \\ \left(\bigotimes_{i=1}^a G_{X_i} \right) \times \left(\left(\bigotimes_{j=1}^b G_{Y_j} \right) \wr_{\mathbb{Z}_m} \mathbb{Z} \right) & \xrightarrow[\cong]{\beta} & G \end{array}$$

in which α and β are isomorphisms and $\bar{\lambda}$ is defined by

$$\bar{\lambda}(\mathbf{h}_1, \dots, \mathbf{h}_a, \xi, k) = \left(\lambda_1(\mathbf{h}_1), \dots, \lambda_a(\mathbf{h}_a), \left(\bigotimes_{j=1}^b \lambda_j \right) \circ \xi, k \bmod m \right),$$

for $\mathbf{h} \in \pi_0 \mathcal{S}_{X_i}$, a map $\xi : \mathbb{Z}_m \rightarrow \bigotimes_{j=1}^b \pi_0 \mathcal{S}_{Y_j}$, and $k \in \mathbb{Z}$. Hence by (3) and (5) of Lemma 2.2 λ arises from a free action on a torus.

Proof. Isomorphisms α and β are constructed in [10] and [9] respectively. Again we will just recall their definitions.

Let $\mathbf{h}_i \in \pi_0 \mathcal{S}_{X_i}$, $i = 1, \dots, a$, $\xi = (\xi_1, \dots, \xi_b) : \mathbb{Z}_m \rightarrow \bigotimes_{j=1}^b \pi_0 \mathcal{S}_{Y_j}$ be any map, and $k \in \mathbb{Z}$. Choose a representative $h_i \in \mathcal{S}_{X_i}$ of \mathbf{h}_i fixed on some neighbourhood of \hat{X}_i . Let also h_{js} , $1 \leq j \leq b$, $s \in \mathbb{Z}_m$, be a representative of $\xi_j(s) \in \pi_0 \mathcal{S}_{Y_j}$ fixed on some

neighbourhood of $\hat{Y}_j = Y_j \cap N$, and $h : M \rightarrow M$ be a map given by the formula:

$$h = \begin{cases} h_i, & \text{on } X_i, \ i = 1, \dots, a \\ g^{s+k} \circ h_{j_s} \circ g^{-s}, & \text{on } g^s(Y_j), \text{ for } s \in \mathbb{Z}_m \text{ and } j = 1, \dots, b, \\ g^k, & \text{on } N \cup Z. \end{cases}$$

Then $h \in \mathcal{S}$ and we define $\alpha(\mathbf{h}_1, \dots, \mathbf{h}_a, \xi, k) \in \pi_0 \mathcal{S}$ to be the isotopy class of h . It is shows in [10] that α is a group isomorphism.

Let $\gamma_i \in G_{X_i}$, $i = 1, \dots, a$, $\nu = (\nu_1, \dots, \nu_b) : \mathbb{Z}_m \rightarrow \bigtimes_{j=1}^b G_{Y_j}$ be map, and $k \in \mathbb{Z}_m$.

Define $\gamma : \Gamma \rightarrow \Gamma$ by the following formula:

$$\gamma = \begin{cases} \gamma_i, & \text{on } \Gamma_{X_i}, \ i = 1, \dots, a \\ \lambda(g)^{s+k} \circ \nu_j(s) \circ \lambda(g)^{-s}, & \text{on } \lambda(g)^s(\Gamma_{Y_j}), \text{ for } s \in \mathbb{Z}_m \text{ and } j = 1, \dots, b, \\ \lambda(g)^k, & \text{on the remained part of } \Gamma. \end{cases}$$

Then $\gamma \in G$ and the correspondence $\beta : (\gamma_1, \dots, \gamma_n) \mapsto \gamma$ is a group isomorphism, see [9]. It also easily follows from the definitions that the diagram (4.2) is commutative. \square

General case. Suppose M is orientable and distinct from S^2 , T^2 , D^2 , and $S^1 \times I$. Set $V = \partial M$ and denote $\mathcal{S} = \mathcal{S}'(f, V)$. Then $\chi(M) < 0$ whence by (1.2) the boundary map $\partial : \pi_1 \mathcal{O}_f(f, V) \rightarrow \pi_0 \mathcal{S}$ is an isomorphism.

Lemma 4.4. [6, 9] *If $\chi(M) < 0$, then there exist mutually disjoint subsurfaces B_1, \dots, B_n of M having the following properties:*

(a) B_i is diffeomorphic either to a 2-disk D^2 or to a cylinder $S^1 \times I$, and the restriction $f_i := f|_{B_i} : B_i \rightarrow P$ is Morse for all $i = 1, \dots, n$.

(b) Let Γ_i be the Kronrod-Reeb graph of f_i , $\mathcal{S}_i = \mathcal{S}'(f_i, \partial B_i)$, $\lambda_i : \pi_0 \mathcal{S}_i \rightarrow \text{Aut}(\Gamma_i)$ be the corresponding action homomorphism, and $G_i = \lambda_i(\mathcal{S}_i)$, $i = 1, \dots, n$. Then we have the following commutative diagram:

$$(4.3) \quad \begin{array}{ccc} \bigtimes_{i=1}^n \pi_0 \mathcal{S}_i & \xrightarrow[\cong]{\alpha} & \pi_0 \mathcal{S} \\ \downarrow \bigtimes_{i=1}^n \lambda_i & & \downarrow \lambda \\ \bigtimes_{i=1}^n G_i & \xrightarrow[\cong]{\beta} & G \end{array}$$

in which α and β are isomorphisms.

Proof. Surfaces $\{B_i\}_{i=1}^n$ and the isomorphism α are defined in [6, Theorem 1.7]. The isomorphism β and diagram (4.3) are constructed in [9]. In fact α and β are similar to the corresponding isomorphisms from Lemma 4.2 and therefore we skip their descriptions. \square

Since each B_i is either a 2-disk D^2 or a cylinder $S^1 \times I$, the homomorphism λ_i arises from a free action of G_i on some torus. Then by (3) of Lemma 2.2 so does $\lambda : \pi_0 \mathcal{S} \rightarrow G$. Proposition 3.1 is completed.

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